積分法　演習プリント　No.2

１．［クリアー数学Ⅲ 問題283(3)(4)　問題284(1)(3)］

次の定積分を求めよ。

(1) $\displaystyle\int_0^{\frac{1}{3}}\frac{dx}{(3x+1)^2}$　　(2) $\displaystyle\int_3^5\frac{dx}{\sqrt{2x-1}}$

(3) $\displaystyle\int_0^1\frac{x^2+x+1}{x+1}dx$　　(4) $\displaystyle\int_{-1}^1\frac{dx}{x^2-5x+6}$

(1) $$\int_0^{\frac{1}{3}}\frac{dx}{(3x+1)^2}=\int_0^{\frac{1}{3}}(3x+1)^{-2}dx=\left[\frac{1}{3}\cdot\{-(3x+1)^{-1}\}\right]_0^{\frac{1}{3}}$$
$$=-\frac{1}{3}\left[\frac{1}{3x+1}\right]_0^{\frac{1}{3}}=-\frac{1}{3}\left(\frac{1}{2}-1\right)=\frac{1}{6}$$

(2) $$\int_3^5\frac{dx}{\sqrt{2x-1}}=\int_3^5(2x-1)^{-\frac{1}{2}}dx=\left[\frac{1}{2}\cdot\left\{2(2x-1)^{\frac{1}{2}}\right\}\right]_3^5$$
$$=\left[\sqrt{2x-1}\right]_3^5=3-\sqrt{5}$$

(3) $$\int_0^1\frac{x^2+x+1}{x+1}dx=\int_0^1\left(x+\frac{1}{x+1}\right)dx=\left[\frac{x^2}{2}+\log|x+1|\right]_0^1=\frac{1}{2}+\log 2$$

(4) $$\int_{-1}^1\frac{dx}{x^2-5x+6}=\int_{-1}^1\frac{dx}{(x-2)(x-3)}=\int_{-1}^1\left(\frac{1}{x-3}-\frac{1}{x-2}\right)dx$$
$$=\Big[\log|x-3|-\log|x-2|\Big]_{-1}^1=\left[\log\left|\frac{x-3}{x-2}\right|\right]_{-1}^1$$
$$=\log 2-\log\frac{4}{3}=\log\frac{3}{2}$$

２．［クリアー数学Ⅲ 問題285(2)(3)(4)(5)　問題292(2)］

次の定積分を求めよ。

(1) $\displaystyle\int_0^{\frac{\pi}{2}}\sin\frac{5}{2}x\sin\frac{x}{2}dx$　　(2) $\displaystyle\int_0^{\frac{\pi}{4}}\cos^2 x\,dx$

(3) $\displaystyle\int_{-\frac{\pi}{2}}^{\pi}\sin^2 2x\,dx$　　(4) $\displaystyle\int_0^{\frac{\pi}{3}}\tan^2 x\,dx$　　(5) $\displaystyle\int_0^{\frac{\pi}{2}}\frac{\sin x}{2-\cos x}dx$

(1) $$\int_0^{\frac{\pi}{2}}\sin\frac{5}{2}x\sin\frac{x}{2}dx=-\frac{1}{2}\int_0^{\frac{\pi}{2}}(\cos 3x-\cos 2x)dx=-\frac{1}{2}\left[\frac{\sin 3x}{3}-\frac{\sin 2x}{2}\right]_0^{\frac{\pi}{2}}$$
$$=-\frac{1}{2}\left(-\frac{1}{3}-0\right)=\frac{1}{6}$$

(2) $$\int_0^{\frac{\pi}{4}}\cos^2 x\,dx=\int_0^{\frac{\pi}{4}}\frac{1+\cos 2x}{2}dx=\frac{1}{2}\left[x+\frac{\sin 2x}{2}\right]_0^{\frac{\pi}{4}}=\frac{1}{2}\left(\frac{\pi}{4}+\frac{1}{2}\right)=\frac{\pi}{8}+\frac{1}{4}$$

(3) $$\int_{-\frac{\pi}{2}}^{\pi}\sin^2 2x\,dx=\int_{-\frac{\pi}{2}}^{\pi}\frac{1-\cos 4x}{2}dx=\frac{1}{2}\left[x-\frac{\sin 4x}{4}\right]_{-\frac{\pi}{2}}^{\pi}=\frac{1}{2}\left\{\pi-\left(-\frac{\pi}{2}\right)\right\}=\frac{3}{4}\pi$$

(4) $$\int_0^{\frac{\pi}{3}}\tan^2 x\,dx=\int_0^{\frac{\pi}{3}}\left(\frac{1}{\cos^2 x}-1\right)dx=\Big[\tan x-x\Big]_0^{\frac{\pi}{3}}=\sqrt{3}-\frac{\pi}{3}$$

(5) $$\int_0^{\frac{\pi}{2}}\frac{\sin x}{2-\cos x}dx=\int_0^{\frac{\pi}{2}}\frac{(2-\cos x)'}{2-\cos x}dx=\Big[\log|2-\cos x|\Big]_0^{\frac{\pi}{2}}=\log 2$$

３．［クリアー数学Ⅲ 問題291］

次の定積分を求めよ。

(1) $\displaystyle\int_1^2 x(x^2-1)^3dx$　　(2) $\displaystyle\int_{-1}^0 x\sqrt{x+1}\,dx$　　(3) $\displaystyle\int_{-\sqrt{3}}^0\frac{2x}{\sqrt{4-x^2}}dx$

(1) $x^2-1=t$ とおくと　　$2xdx=dt$

x	$1\longrightarrow 2$
t	$0\longrightarrow 3$

よって
$$\int_1^2 x(x^2-1)^3dx=\frac{1}{2}\int_1^2 2x(x^2-1)^3dx$$
$$=\frac{1}{2}\int_0^3 t^3dt=\frac{1}{2}\left[\frac{t^4}{4}\right]_0^3=\frac{81}{8}$$

(2) $\sqrt{x+1}=t$ とおくと　　$x=t^2-1$，$dx=2tdt$

x	$-1\longrightarrow 0$
t	$0\longrightarrow 1$

よって
$$\int_{-1}^0 x\sqrt{x+1}\,dx=\int_0^1(t^2-1)t\cdot 2tdt$$
$$=2\int_0^1(t^4-t^2)dt=2\left[\frac{t^5}{5}-\frac{t^3}{3}\right]_0^1$$
$$=2\left(\frac{1}{5}-\frac{1}{3}\right)=-\frac{4}{15}$$

(3) $4-x^2=t$ とおくと　　$-2xdx=dt$

x	$-\sqrt{3}\longrightarrow 0$
t	$1\longrightarrow 4$

よって
$$\int_{-\sqrt{3}}^0\frac{2x}{\sqrt{4-x^2}}dx=-\int_1^4\frac{dt}{\sqrt{t}}=-\left[2t^{\frac{1}{2}}\right]_1^4$$
$$=-2\Big[\sqrt{t}\Big]_1^4=-2(2-1)=-2$$

４．［クリアー数学Ⅲ 問題293］

次の定積分を求めよ。

(1) $\displaystyle\int_0^3\sqrt{9-x^2}\,dx$　　(2) $\displaystyle\int_{-\frac{\sqrt{2}}{2}}^1\sqrt{2-x^2}\,dx$　　(3) $\displaystyle\int_{-1}^1\frac{dx}{\sqrt{4-x^2}}$

(1) $x=3\sin\theta$ とおくと　　$dx=3\cos\theta\,d\theta$

x と θ の対応は右のようにとれる。

x	$0\longrightarrow 3$
θ	$0\longrightarrow\frac{\pi}{2}$

また，$0\leqq\theta\leqq\frac{\pi}{2}$ のとき $\cos\theta\geqq 0$ であるから
$$\sqrt{9-x^2}=\sqrt{9(1-\sin^2\theta)}$$
$$=\sqrt{9\cos^2\theta}=3\cos\theta$$

よって
$$\int_0^3\sqrt{9-x^2}\,dx=\int_0^{\frac{\pi}{2}}(3\cos\theta)\cdot 3\cos\theta\,d\theta$$
$$=9\int_0^{\frac{\pi}{2}}\cos^2\theta\,d\theta=9\int_0^{\frac{\pi}{2}}\frac{1+\cos 2\theta}{2}d\theta$$
$$=\frac{9}{2}\left[\theta+\frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{2}}=\frac{9}{4}\pi$$

参考　求める定積分の値は，半径 3 の円の面積の $\frac{1}{4}$ であるから　　$\frac{1}{4}\pi\cdot 3^2=\frac{9}{4}\pi$

(2) $x=\sqrt{2}\sin\theta$ とおくと　　$dx=\sqrt{2}\cos\theta\,d\theta$

x と θ の対応は右のようにとれる。

x	$-\frac{\sqrt{2}}{2}\longrightarrow 1$
θ	$-\frac{\pi}{6}\longrightarrow\frac{\pi}{4}$

また，$-\frac{\pi}{6}\leqq\theta\leqq\frac{\pi}{4}$ のとき $\cos\theta\geqq 0$ であるから
$$\sqrt{2-x^2}=\sqrt{2(1-\sin^2\theta)}$$
$$=\sqrt{2\cos^2\theta}=\sqrt{2}\cos\theta$$

よって
$$\int_{-\frac{\sqrt{2}}{2}}^1\sqrt{2-x^2}\,dx=\int_{-\frac{\pi}{6}}^{\frac{\pi}{4}}(\sqrt{2}\cos\theta)\cdot\sqrt{2}\cos\theta\,d\theta$$
$$=2\int_{-\frac{\pi}{6}}^{\frac{\pi}{4}}\cos^2\theta\,d\theta=2\int_{-\frac{\pi}{6}}^{\frac{\pi}{4}}\frac{1+\cos 2\theta}{2}d\theta$$
$$=\left[\theta+\frac{\sin 2\theta}{2}\right]_{-\frac{\pi}{6}}^{\frac{\pi}{4}}=\frac{5}{12}\pi+\frac{1}{2}+\frac{\sqrt{3}}{4}$$

(3) $x=2\sin\theta$ とおくと　　$dx=2\cos\theta\,d\theta$

x と θ の対応は右のようにとれる。

x	$-1\longrightarrow 1$
θ	$-\frac{\pi}{6}\longrightarrow\frac{\pi}{6}$

また，$-\frac{\pi}{6}\leqq\theta\leqq\frac{\pi}{6}$ のとき $\cos\theta\geqq 0$ であるから
$$\sqrt{4-x^2}=\sqrt{4(1-\sin^2\theta)}$$
$$=\sqrt{4\cos^2\theta}=2\cos\theta$$

よって
$$\int_{-1}^1\frac{dx}{\sqrt{4-x^2}}=\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}}\frac{2\cos\theta}{2\cos\theta}d\theta=\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}}d\theta=\Big[\theta\Big]_{-\frac{\pi}{6}}^{\frac{\pi}{6}}=\frac{\pi}{3}$$

積分法　演習プリント　No.2

５．［クリアー数学Ⅲ 問題294］

次の定積分を求めよ。

(1) $\displaystyle\int_0^{2\sqrt{3}}\frac{dx}{x^2+4}$　　(2) $\displaystyle\int_{\sqrt{3}}^{3\sqrt{3}}\frac{dx}{x^2+9}$　　(3) $\displaystyle\int_{\sqrt{2}}^{\sqrt{6}}\frac{dx}{3x^2+6}$

(1) $x=2\tan\theta$ とおくと　$dx=\dfrac{2}{\cos^2\theta}d\theta$

x と θ の対応は右のようにとれる。

x	$0 \longrightarrow 2\sqrt{3}$
θ	$0 \longrightarrow \dfrac{\pi}{3}$

よって

$$\int_0^{2\sqrt{3}}\frac{dx}{x^2+4}=\int_0^{\frac{\pi}{3}}\frac{1}{4(\tan^2\theta+1)}\cdot\frac{2}{\cos^2\theta}d\theta$$

$$=\int_0^{\frac{\pi}{3}}\frac{\cos^2\theta}{4}\cdot\frac{2}{\cos^2\theta}d\theta$$

$$=\frac{1}{2}\int_0^{\frac{\pi}{3}}d\theta=\frac{1}{2}\Big[\theta\Big]_0^{\frac{\pi}{3}}=\frac{\pi}{6}$$

(2) $x=3\tan\theta$ とおくと　$dx=\dfrac{3}{\cos^2\theta}d\theta$

x と θ の対応は右のようにとれる。

x	$\sqrt{3} \longrightarrow 3\sqrt{3}$
θ	$\dfrac{\pi}{6} \longrightarrow \dfrac{\pi}{3}$

よって

$$\int_{\sqrt{3}}^{3\sqrt{3}}\frac{dx}{x^2+9}=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{1}{9(\tan^2\theta+1)}\cdot\frac{3}{\cos^2\theta}d\theta$$

$$=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{\cos^2\theta}{9}\cdot\frac{3}{\cos^2\theta}d\theta$$

$$=\frac{1}{3}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}d\theta=\frac{1}{3}\Big[\theta\Big]_{\frac{\pi}{6}}^{\frac{\pi}{3}}=\frac{\pi}{18}$$

(3) $\displaystyle\int_{\sqrt{2}}^{\sqrt{6}}\frac{dx}{3x^2+6}=\frac{1}{3}\int_{\sqrt{2}}^{\sqrt{6}}\frac{dx}{x^2+2}$

$x=\sqrt{2}\tan\theta$ とおくと　$dx=\dfrac{\sqrt{2}}{\cos^2\theta}d\theta$

x と θ の対応は右のようにとれる。

x	$\sqrt{2} \longrightarrow \sqrt{6}$
θ	$\dfrac{\pi}{4} \longrightarrow \dfrac{\pi}{3}$

よって

$$\int_{\sqrt{2}}^{\sqrt{6}}\frac{dx}{3x^2+6}=\frac{1}{3}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{1}{2(\tan^2\theta+1)}\cdot\frac{\sqrt{2}}{\cos^2\theta}d\theta$$

$$=\frac{1}{3}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{\cos^2\theta}{2}\cdot\frac{\sqrt{2}}{\cos^2\theta}d\theta$$

$$=\frac{\sqrt{2}}{6}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}d\theta=\frac{\sqrt{2}}{6}\Big[\theta\Big]_{\frac{\pi}{4}}^{\frac{\pi}{3}}=\frac{\sqrt{2}}{72}\pi$$

６．［クリアー数学Ⅲ 問題301 (1)(5)(6)　問題303 (2)］

次の定積分を求めよ。

(1) $\displaystyle\int_0^{\frac{\pi}{2}}x\cos 3x\,dx$　　(2) $\displaystyle\int_{-4}^{-3}\log(x+5)\,dx$

(3) $\displaystyle\int_1^{e}x^2\log x\,dx$　　(4) $\displaystyle\int_{-1}^{1}x^2e^{2x}dx$

(1) $$\int_0^{\frac{\pi}{2}}x\cos 3x\,dx=\int_0^{\frac{\pi}{2}}x\left(\frac{\sin 3x}{3}\right)'dx=\left[\frac{x\sin 3x}{3}\right]_0^{\frac{\pi}{2}}-\int_0^{\frac{\pi}{2}}\frac{\sin 3x}{3}dx$$

$$=-\frac{\pi}{6}+\left[\frac{\cos 3x}{9}\right]_0^{\frac{\pi}{2}}=-\frac{\pi}{6}-\frac{1}{9}$$

(2) $$\int_{-4}^{-3}\log(x+5)\,dx=\int_{-4}^{-3}(x+5)'\log(x+5)\,dx$$

$$=\Big[(x+5)\log(x+5)\Big]_{-4}^{-3}-\int_{-4}^{-3}(x+5)\cdot\frac{1}{x+5}dx$$

$$=2\log 2-\Big[x\Big]_{-4}^{-3}=2\log 2-1$$

(3) $$\int_1^{e}x^2\log x\,dx=\int_1^{e}\left(\frac{x^3}{3}\right)'\log x\,dx$$

$$=\left[\frac{x^3}{3}\log x\right]_1^{e}-\int_1^{e}\frac{x^3}{3}\cdot\frac{1}{x}dx=\frac{e^3}{3}-\int_1^{e}\frac{x^2}{3}dx$$

$$=\frac{e^3}{3}-\left[\frac{x^3}{9}\right]_1^{e}=\frac{e^3}{3}-\frac{1}{9}(e^3-1)=\frac{2}{9}e^3+\frac{1}{9}$$

(4) $$\int_{-1}^{1}x^2e^{2x}dx=\int_{-1}^{1}x^2\left(\frac{e^{2x}}{2}\right)'dx=\left[\frac{x^2e^{2x}}{2}\right]_{-1}^{1}-\int_{-1}^{1}xe^{2x}dx$$

$$=\frac{e^2}{2}-\frac{1}{2e^2}-\int_{-1}^{1}x\left(\frac{e^{2x}}{2}\right)'dx$$

$$=\frac{e^2}{2}-\frac{1}{2e^2}-\left(\left[\frac{xe^{2x}}{2}\right]_{-1}^{1}-\int_{-1}^{1}\frac{e^{2x}}{2}dx\right)$$

$$=\frac{e^2}{2}-\frac{1}{2e^2}-\left(\frac{e^2}{2}+\frac{1}{2e^2}\right)+\left[\frac{e^{2x}}{4}\right]_{-1}^{1}$$

$$=-\frac{1}{e^2}+\left(\frac{e^2}{4}-\frac{1}{4e^2}\right)=\frac{e^2}{4}-\frac{5}{4e^2}$$